Mathematical Induction

- Very important method of proof in mathematics
- Nothing to do with inductive reasoning which draws conclusions from a large number of special cases but does not actually offer a certain conclusion.
- Is deductive in nature leading to a firm conclusion.
- Usually applied in proving validity of statement involving all positive integral values of n.

Mathematical Induction

Assume have ladder with indefinite number of steps and wish to proof that can climb to any designated step.

Can do this if know two things:

(a) Can climb to first step,

(b) If on any step, can climb to next higher step.

From (a) know can get on first step. With this knowledge and fact stated in (b) know can get on second step. Again with this knowledge and the fact stated in (b), we know we can get on the third step, etc.

Induction proofs have four components:

(1) the thing you want to prove, (Write out the formula to be proved)

(2) the beginning step (usually "let n = 1"), (Check that true for n = 1)

(3) the assumption step ("let n = k"), and (Assume true for n=k and write formula replacing n with k.)

(4) the induction step ("let n = k + 1"). (Replace k with k+1 and see if both sides of formula still balance out)

For example, say you notice that when you add up all the numbers from 1 to n (i.e.1 + 2 + 3 + 4 + ... + n), you get a total = (n)(n+1)/2. For every number that you've checked so far, you get

$$1 + 2 + 3 + 4 + ... + n = (n)(n+1)/2$$

For convenience, call this formula "(*)", or "star". You'd like to know if (*) is true for all whole numbers, but obviously can't check every single whole number, as they go on for ever. You only know that it's true for the relatively few numbers that you've actually checked. Adding all those numbers soon becomes tedious, so you'd really like for (*) to work. But to use (*) for whatever number you'd like, you somehow have to prove that (*) works everywhere. Induction proofs allow you to prove that the formula works "everywhere" without your having to actually show that it works everywhere (by doing the infinitely-many additions).

So you have the first part of an induction proof, the formula that you'd like to prove:

(*) For all natural numbers n, 1 + 2 + 3 + 4 + ... + n = (n)(n+1)/2

But is (*) ever true anywhere? If you can't find any particular number, for which (*) is known to be true, then there's really no point in continuing. So you do the second part, which is to show that (*) is indeed true for some particular number:

Let n = 1. Then 1 + 2 + 3 + 4 + ... + n is actually just 1; that is:

(n)(n+1)/2 = (1)(1+1)/2 = (1)(2)/2 = 1. We get 1 = 1, so (*) is true at n = 1.

We've shown that (*) works in some particular place. But does it work anywhere else? Well, that was our problem in the first place: we know that it works in a few places, but we need to prove that (*) works everywhere, and we'll never do that by proving one case at a time. So we need the third and fourth parts of the induction proof.

The third part is the assumption part: Let n = k.

Assume that, for n = k, (*) works; that is, assume that: 1 + 2 + 3 + 4 + ... + k = (k)(k+1)/2

This is not the same as the second part, where we named the actual place where (*) worked. This part just says "Let's assume that (*) works, somewhere, I'm not saying where, I might not even know where; just somewhere. In the fourth step, if we can prove, assuming (*) works at n = k, that (*) then works at n = k + 1 (that is, if it works some place, then it must also work at the next place), then, since we know of a certain place (n = 1) where (*) works, we will have proved that (*) works everywhere:

Let n = k + 1.
Then 1 + 2 + 3 + 4 + ... + k + (k + 1) [the left-hand side of (*)]
  = [1 + 2 + 3 + 4 + ... + k] + k + 1 [from part three]
  = [(k)(k+1)/2] + k + 1 [our assumption]
  = [(k)(k+1)/2] + 2(k+1)/2 [common denominator]
  = (k)(k+1)+2(k+1)/2 [adding fractions]
  = (k+2)(k+1)/2 [simplifying]
  = ((k+1)+1)(k+1)/2 [restating in "k + 1" form]

This last line is the right-hand side of (*). In other words, if we assume that (*) works as some unnamed number k, then we can show (by using that assumption) that (*) works at the next number, k + 1. And we already know of a number where (*) works! Since we showed that (*) works at n = 1, the assumption and induction steps tell us that (*) then works at n = 2, and then by induction (*) works at n = 3, and then by induction, (*) works at n = 4, and so on. The assumption and induction steps allow us to make the jump from "It works here and there" to "It works everywhere!" It's like dominoes: instead of knocking each one down individually, you say "If I can knock down one of them, then that one will knock down the next one, and then the next one, and eventually all of them; and – look! – I can knock down this first one right here."

Question 4 **(25 marks)**

(a) Prove, by induction, the formula for the sum of the first n terms of a geometric series. That is, prove that, for $r \neq 1$:

$$a + ar + ar^2 + \cdots + ar^{n-1} = \frac{a(1-r^n)}{1-r}.$$

$P(n)$: $\quad a + ar + ar^2 + \cdots + ar^{n-1} = \dfrac{a(1-r^n)}{1-r}$

Check $P(1)$: $\quad a = \dfrac{a(1-r)}{1-r}$, which is true.

Assume $P(k)$: $\quad a + ar + ar^2 + \cdots + ar^{k-1} = \dfrac{a(1-r^k)}{1-r}$

Then:

$$\underbrace{a + ar + ar^2 + \cdots + ar^{k-1}} + ar^k$$

$$= \frac{a(1-r^k)}{1-r} + ar^k$$

$$= \frac{a(1-r^k) + ar^k(1-r)}{1-r}$$

$$= \frac{a(1-\cancel{r^k}+\cancel{r^k}-r^{k+1})}{1-r}$$

$$= \frac{a(1-r^{k+1})}{1-r}$$

which establishes $P(k+1)$.

Since we have $P(1) \wedge \{\forall k \in \mathbb{N}, (P(k) \Rightarrow P(k+1)\}$, it follows that $P(n)$ holds $\forall n \in \mathbb{N}$.

De Moivre's Theorem says:

$$(\cos(\theta) + i\sin(\theta))^n = \cos n\theta + i\sin n\theta$$

Proof of De Moivre's Theorem by Induction

We just show De Moivre for positive integer values of n. Remember we need our starting step for induction proofs. For $n = 1$ we have:

$$(\cos(\theta) + i\sin(\theta))^1 = \cos 1\theta + i\sin 1\theta.$$

So De Moivre holds for $n = 1$. Assume now it's true for $n = k$. I.e.:

$$(\cos(\theta) + i\sin(\theta))^k = \cos k\theta + i\sin k\theta. \tag{1}$$

We want to show it is also true for $n = k + 1$, and then we'll be done. Consider then:

$$(\cos(\theta) + i\sin(\theta))^{k+1} = (\cos(\theta) + i\sin(\theta))(\cos(\theta) + i\sin(\theta))^k.$$

We now use equation (1), and sub in for $(\cos(\theta) + i\sin(\theta))^k$ on the right hand side. Then we get

$$(\cos(\theta) + i\sin(\theta))^{k+1} = (\cos(\theta) + i\sin(\theta))(\cos(k\theta) + i\sin(k\theta)).$$

$$= \cos(\theta)\cos(k\theta) + i^2\sin(\theta)\sin(k\theta) + i\sin(\theta)\cos(k\theta) + i\sin(k\theta)\cos(\theta)$$

$$= \cos(\theta)\cos(k\theta) \quad \sin(\theta)\sin(k\theta) + i(\sin(\theta)\cos(k\theta) + \sin(k\theta)\cos(\theta))$$

after expanding the brackets. Now, as you all know (without having to look it up in your formula booklet, of course), $\sin(a + b) = \sin a\cos b + \sin b\cos a$ and $\cos(a + b) = \cos a\cos b \quad \sin a\sin b$. So we have

$$(\cos(\theta) + i\sin(\theta))^{k+1} = \cos(k + 1)\theta + i\sin(k + 1)\theta.$$

But this is just what we need, right? So by induction we've shown De Moivre to be true for all positive integers n.

Binomial Theorem

For any positive integer n,

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k.$$

Proof by Induction:

For $n = 1$,

$$(x+y)^1 = x + y = \binom{1}{0} x^{1-0} y^0 + \binom{1}{1} x^{1-1} y^1 = \sum_{k=0}^{1} \binom{1}{k} x^{1-k} y^k.$$

Suppose $(x+y)^{n-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} x^{(n-1)-k} y^k$.

Consider $(x+y)^n$.

$$\begin{aligned}
(x+y)^n &= (x+y)(x+y)^{n-1} \\
&= (x+y)\left[\sum_{k=0}^{n-1} \binom{n-1}{k} x^{(n-1)-k} y^k\right] \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{j=0}^{n-1} \binom{n-1}{j} x^{(n-1)-j} y^{j+1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{j=0}^{n-1} \binom{n-1}{(j+1)-1} x^{n-(j+1)} y^{j+1} \\
&= \sum_{k=0}^{n-1} \binom{n-1}{k} x^{n-k} y^k + \sum_{k=1}^{n} \binom{n-1}{k-1} x^{n-k} y^k \\
&= \sum_{k=0}^{n} \left[\binom{n-1}{k} x^{n-k} y^k\right] - \binom{n-1}{n} x^0 y^n \\
&\quad + \sum_{k=0}^{n} \left[\binom{n-1}{k-1} x^{n-k} y^k\right] - \binom{n-1}{-1} x^n y^0 \\
&= \sum_{k=0}^{n} \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] x^{n-k} y^k \\
&= \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k
\end{aligned}$$

Prove by induction that, for all $n \in \mathbb{Z}_+$, $\sum_{i=1}^{n}(-1)^i i^2 = (-1)^n n(n+1)/2$.

Proof: *We will prove by induction that, for all* $n \in \mathbb{Z}_+$,

$$\sum_{i=1}^{n}(-1)^i i^2 = \frac{(-1)^n n(n+1)}{2}. \tag{1}$$

Base case: *When* $n = 1$, *the left side of (1) is* $(-1)1^2 = -1$, *and the right side is* $(-1)1(1+1)/2 = -1$, *so both sides are equal and (1) is true for* $n = 1$.

Induction step: *Let* $k \in \mathbb{Z}_+$ *be given and suppose (1) is true for* $n = k$. *Then*

$$\begin{aligned}
\sum_{i=1}^{k+1}(-1)^i i^2 &= \sum_{i=1}^{k}(-1)^i i^2 + (-1)^{k+1}(k+1)^2 \\
&= \frac{(-1)^k k(k+1)}{2} + (-1)^{k+1}(k+1)^2 \quad \textit{(by induction hypothesis)} \\
&= \frac{(-1)^k (k+1)}{2}\left(k - 2(k+1)\right) \\
&= \frac{(-1)^k (k+1)}{2}\left(-k - 2\right) \\
&= \frac{(-1)^{k+1}(k+2)}{2}.
\end{aligned}$$

Thus, (1) holds for $n = k + 1$, *and the proof of the induction step is complete.*

Conclusion: *By the principle of induction, (1) is true for all* $n \in \mathbb{Z}_+$.

Prove that $n! > 2^n$ for $n \geq 4$.

Proof: *We will prove by induction that*

$$(*) \qquad n! > 2^n$$

holds for all $n \geq 4$.

Base case: *Our base case here is the first* n*-value for which* $(*)$ *is claimed, i.e.,* $n = 4$. *For* $n = 4$, *the left and right sides of* $(*)$ *are 24 and 16, respectively, so* $(*)$ *is true in this case.*

Induction step: *Let* $k \geq 4$ *be given and suppose* $(*)$ *is true for* $n = k$. *Then*

$$\begin{aligned} (k+1)! &= k!(k+1) \\ &> 2^k(k+1) \quad \text{(by induction hypothesis)} \\ &\geq 2^k \cdot 2 \quad \text{(since } k \geq 4 \text{ and so } k+1 \geq 2\text{))} \\ &= 2^{k+1}. \end{aligned}$$

Thus, $(*)$ *holds for* $n = k+1$, *and the proof of the induction step is complete.*

Conclusion: *By the principle of induction, it follows that* $(*)$ *is true for all* $n \geq 4$.

You might feel uneasy about that assumption step. If we assume something is true, can't we prove anything?

Well, actually, no. But your text probably gives a list of formulas for you to "prove..." or "determine whether true...", and they all turn out to be true. So, in spite of good intentions, your teacher and your book have been quite misleading. To help you feel more confident about induction, let's try to prove a couple of statements that we know are wrong, so you can see that you can't use induction to prove something that ain't so.

Take (*) For all n > 0, n3 < n2

We know that this statement isn't so: cubes of whole numbers are bigger than squares (except for the cube and square of 1). But let's try to prove this false statement, and see what happens.

Let n = 1. Then n3 = 13 = 1 and n2 = 12 = 1, and 1 < 1.

Then (*) holds for n = 1.

The first part worked, but let's continue, and see what happens:

Let n = k.

Assume that, for n = k, we have k3 < k2.

For this next bit, just trust me: everything will become clear at the end:

Let n = k + 1.

Whatever k is, we know that three of their square is bigger than just one of their square; that is, 3k2 > k2, because k2 > 0.

Also, we know that three of the value is bigger than two; that is, 3k > 2k, because k > 0.

Also, we know that k3 > 0, because k > 0.

Also, 1 > 1.

Adding together all the left-hand sides and all the right-hand sides above, we get:

k3 + 3k2 + 3k + 1 > k2 + 2k + 1

..which can be restated as: (k + 1)3 > (k + 1)2

But we needed to prove just the opposite!!
That is, (*) failed the induction step. Even if (*) is true in some one place, it will not be true at the next place. So, even though (*) was true for n = 1, it was not true for n = 2, and (*) fails, as we knew it ought to!

Let's try another one. In this one, we'll do the steps out of order:

(*) For all n, $n + 1 < n$

Obviously this isn't true. But let's see what happens if we assume that it is true somewhere:

Let $n = k$. Assume, for $n = k$, that (*) holds; that is, assume that $k + 1 < k$

Let $n = k + 1$. Then:

$$(k + 1) + 1 < (k) + 1 = (k + 1)$$

...so:

$$(k + 1) + 1 < (k + 1)$$

...and (*) holds for $n = k + 1$.

Well, that's a bit disturbing: if (*) is true anywhere, then it's true everywhere. Ah, but we haven't shown (*) to be true anywhere! And that's where the induction proof fails in this case. You can't find any number for which this (*) is true. Since there is no starting point, then induction fails, just as we knew it ought to.